Cumulative Sum Chart

Dr. Raghu Nandan Sengupta
Professor
Department of Industrial and Management Engineering



All figures are taken from(unless otherwise mentioned):
Introduction to Statistical process Control
Douglas. C Montgomery
6th Edition

Problem with Shewhart charts in Phase 2 monitoring

- A major disadvantage of a Shewhart control chart is that it uses only the information about the process contained in the last sample observation and it ignores any information given by the entire sequence of points. This feature makes the Shewhart control chart relatively insensitive to small process shifts, say, on the order of about 1.5σ or less.
- This potentially makes Shewhart control charts less useful in phase II monitoring problems, where the process tends to operate in control, reliable estimates of the process parameters (such as the mean and standard deviation) are available, and assignable causes do not typically result in large process upsets or disturbances.
- Use of warning limits or similar procdures reduce the simplicity and ease of interpretation of the Shewhart control chart, and as we have previously observed, they also dramatically reduce the average run length of the chart when the process is actually in control. This can be very undesirable in phase II process monitoring.
- Two very effective alternatives to the Shewhart control chart may be used when small process shifts are of interest: the cumulative sum (cusum) control chart, and the exponentially weighted moving average (EWMA) control chart.

The CUSUM Control Chart

- Consider the data given in the table
- The first 20 of these observations were drawn at random from a normal distribution with mean $\mu = 10$ and standard deviation $\sigma = 1$.
- Using these the Shewhart control chart is drawn

Data for the Cusum Example

Sample, i	(a) x_i	(b) $x_i - 10$	(c) $C_i = (x_i - 10)$
1	9.45	−0.55	−0.55
2	7.99	−2.01	−2.56
3	9.29	−0.71	−3.27
4	11.66	1.66	−1.61
5	12.16	2.16	0.55
6	10.18	0.18	0.73
7	8.04	−1.96	−1.23
8	11.46	1.46	0.23
9	9.20	−0.80	−0.57
10	10.34	0.34	−0.23
11	9.03	−0.97	−1.20
12	11.47	1.47	0.27
13	10.51	0.51	0.78
14	9.40	−0.60	0.18
15	10.08	0.08	0.26
16	9.37	−0.63	−0.37
17	10.62	0.62	0.25
18	10.31	0.31	0.56
19	8.52	−1.48	−0.92
20	10.84	0.84	−0.08
21	10.90	0.90	0.82
22	9.33	−0.67	0.15
23	12.29	2.29	2.44
24	11.50	1.50	3.94
25	10.60	0.60	4.54
26	11.08	1.08	5.62
27	10.38	0.38	6.00
28	11.62	1.62	7.62
29	11.31	1.31	8.93
30	10.52	0.52	9.45

The chart

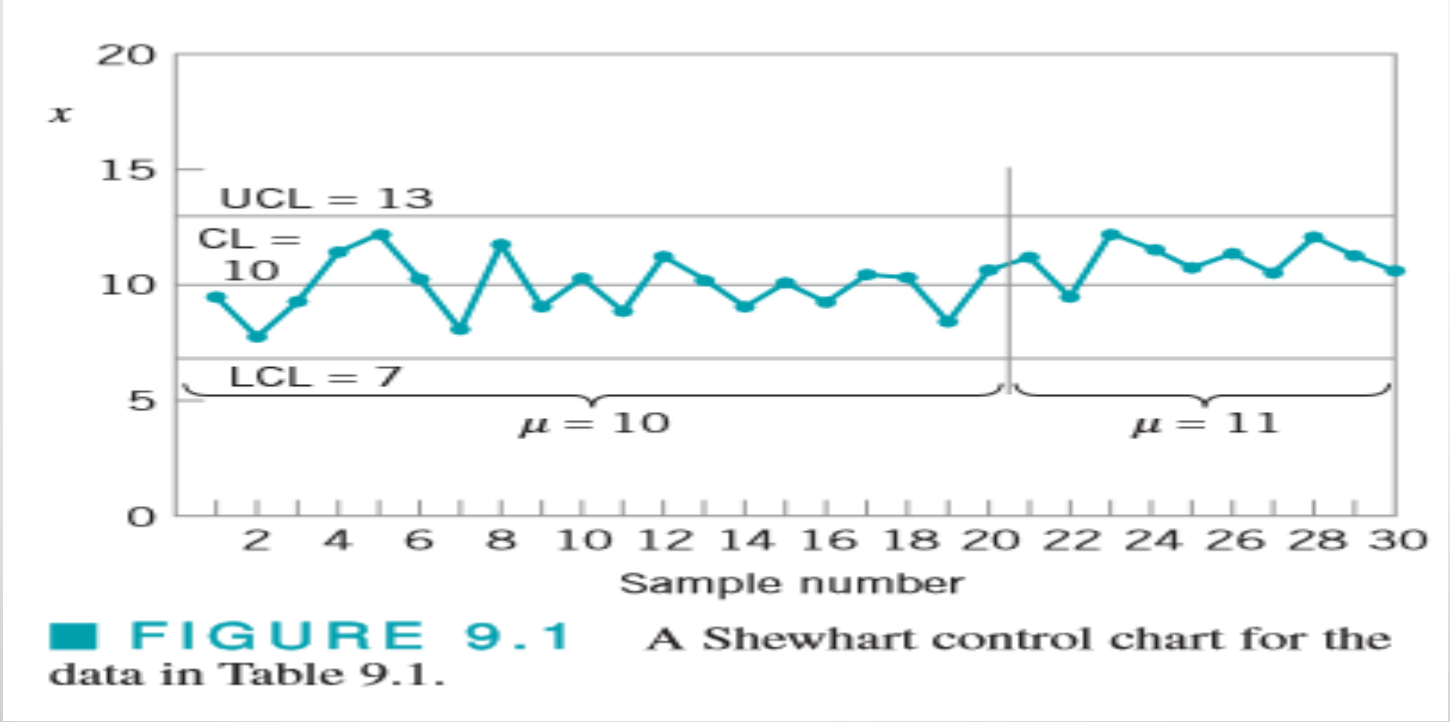


FIGURE 9.1 A Shewhart control chart for the data in Table 9.1.

- Notice that all 20 points are in control, however……

Failure of detecting small shifts

- The last 10 observations in column (a) of Table 9.1 were drawn from a normal distribution with mean m = 11 and standard deviation σ = 1. Consequently, we can think of these last 10 observations as having been drawn from the process when it is out of control—that is, after the process has experienced a shift in the mean of 1σ.
- These last 10 observations are also plotted on the control chart in Fig. 9.1. None of these points plots outside the control limits, so we have no strong evidence that the process is out of control. Note that there is an indication of a shift in process level for the last 10 points, because all but one of the points plot above the center line.
- If we rely on the traditional signal of an out-of-control process, one or more points beyond a three-sigma control limit, then the Shewhart control chart has failed to detect the shift.
- The reason for this failure, of course, is the relatively small magnitude of the shift. The Shewhart chart for averages is very effective if the magnitude of the shift is 1.5σ to 2σ or larger. For smaller shifts, it is not as effective.

Plotting the CUSUM chart

- The cusum chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value.
- Suppose that samples of size n ≥1 are collected, and x_j is the average of the jth sample. Then if μ_0 is the target for the process mean, the cumulative sum control chart is formed by plotting the quantity

$$C_i = \sum_{j=1}^{i} \left(\bar{x}_j - \mu_0 \right)$$

against the sample number i.

Ways to represent CUSUM

- Because they combine information from several samples, cumulative sum charts are more effective than Shewhart charts for detecting small process shifts.
- They are particularly effective with samples of size n=1. This makes the cumulative sum control chart a good candidate for use in the chemical and process industries where rational subgroups are frequently of size 1, and in discrete parts manufacturing with automatic measurement of each part and on-line process monitoring directly at the work center.
- There are two ways to represent cusums, the tabular (or algorithmic) cusum, and the V-mask form of the cusum. Of the two representations, the tabular cusum is preferable. We will not discuss the V-mask procedure in this course.

The Tabular or Algorithmic Cusum for Monitoring the Process Mean

- Cusums may be constructed both for individual observations and for the averages of rational subgroups.
- Let x_i be the ith observation on the process. When the process is in control, x_i has a normal distribution with mean m_0 and standard deviation σ. We assume that either s is known or that a reliable estimate is available. These assumptions are very consistent with phase II applications of SPC.
- Sometimes we think of μ_0 as a target value for the quality characteristic x. This viewpoint is often taken in the chemical and process industries when the objective is to control x (viscosity, say) to a particular target value (such as 2000 centistokes at 100°C).
- If the process drifts or shifts off this target value, the cusum will signal, and an adjustment is made to some manipulatable variable (such as the catalyst feed rate) to bring the process back on target. Also, in some cases a signal from a cusum indicates the presence of an assignable cause that must be investigated just as in the Shewhart chart case.

The cusum values

- The tabular cusum works by accumulating derivations from mo that are above target with one statistic C^+ and accumulating derivations from mo that are below target with another statistic C^-. The statistics C^+ and C^- are called one-sided upper and lower cusums.

The Tabular Cusum

$$C_i^+ = \max\left[0, x_i - (\mu_0 + K) + C_{i-1}^+\right] \quad (9.2)$$

$$C_i^- = \max\left[0, (\mu_0 - K) - x_i + C_{i-1}^-\right] \quad (9.3)$$

where the starting values are $C_0^+ = C_0^- = 0$.

Estimating K and H

- K is usually called the reference value (or the allowance, or the slack value), and it is often chosen about halfway between the target μ_0 and the out-of-control value of the mean μ_1 that we are interested in detecting quickly.
- Considering $\mu_1 = \mu_0 + \delta\sigma$ (or $\delta = |\mu_1 - \mu_0|/\sigma$)
- We can calculate K as $K = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$
- If either C^+_i or C^-_i exceeds the decision interval H, the process is considered to be out of control
  - One rule of thumb to calculate H is to calculate H as 5 times the process standard deviation σ

An example

Set up the tabular cusum using the data from Table 9.1.

Data for the Cusum Example

Sample, i	(a) x_i	(b) $x_i - 10$	(c) $C_i = (x_i - 10)$
1	9.45	−0.55	−0.55
2	7.99	−2.01	−2.56
3	9.29	−0.71	−3.27
4	11.66	1.66	−1.61
5	12.16	2.16	0.55
6	10.18	0.18	0.73
7	8.04	−1.96	−1.23
8	11.46	1.46	0.23
9	9.20	−0.80	−0.57
10	10.34	0.34	−0.23
11	9.03	−0.97	−1.20
12	11.47	1.47	0.27
13	10.51	0.51	0.78
14	9.40	−0.60	0.18
15	10.08	0.08	0.26
16	9.37	−0.63	−0.37
17	10.62	0.62	0.25
18	10.31	0.31	0.56
19	8.52	−1.48	−0.92
20	10.84	0.84	−0.08
21	10.90	0.90	0.82
22	9.33	−0.67	0.15
23	12.29	2.29	2.44
24	11.50	1.50	3.94
25	10.60	0.60	4.54
26	11.08	1.08	5.62
27	10.38	0.38	6.00
28	11.62	1.62	7.62
29	11.31	1.31	8.93
30	10.52	0.52	9.45

Solution

		(a)			(b)		
Period i	x_i	$x_i - 10.5$	C_i^+	N^+	$9.5 - x_i$	C_i^-	N^-
1	9.45	–1.05	0	0	0.05	0.05	1
2	7.99	–2.51	0	0	1.51	1.56	2
3	9.29	–1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	–2.16	0	0
5	12.16	1.66	2.82	2	–2.66	0	0
6	10.18	–0.32	2.50	3	–0.68	0	0
7	8.04	–2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1.00	5	–1.96	0	0
9	9.20	–1.3	0	0	0.30	0.30	1
10	10.34	–0.16	0	0	–0.84	0	0
11	9.03	–1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	–1.97	0	0
13	10.51	0.01	0.98	2	–1.01	0	0
14	9.40	–1.10	0	0	0.10	0.10	1
15	10.08	–0.42	0	0	–0.58	0	0
16	9.37	–1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	–1.12	0	0
18	10.31	–0.19	0	0	–0.81	0	0
19	8.52	–1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	–1.34	0	0
21	10.90	0.40	0.74	2	–1.40	0	0
22	9.33	–1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	–2.79	0	0
24	11.50	1.00	2.79	2	–2.00	0	0
25	10.60	0.10	2.89	3	–1.10	0	0
26	11.08	0.58	3.47	4	–1.58	0	0
27	10.38	–0.12	3.35	5	–0.88	0	0
28	11.62	1.12	4.47	6	–2.12	0	0
29	11.31	0.81	5.28	7	–1.81	0	0
30	10.52	0.02	5.30	8	–1.02	0	0

Solution Explanation…

Recall that the target value is $\mu_0 = 10$, the subgroup size is $n = 1$, the process standard deviation is $\sigma = 1$, and suppose that the magnitude of the shift we are interested in detecting is $1.0\sigma = 1.0(1.0) = 1.0$. Therefore, the out-of-control value of the process mean is $\mu_1 = 10 + 1 = 11$. We will use a tabular cusum with $K = \frac{1}{2}$ (because the shift size is 1.0σ and $\sigma = 1$) and $H = 5$

Table 9.2 presents the tabular cusum scheme. To illustrate the calculations, consider period 1. The equations for C_i^+ and C_i^- are

$$C_1^+ = \max\left[0, x_1 - 10.5 + C_0^+\right]$$

and

$$C_1^- = \max\left[0, 9.5 - x_1 + C_0^-\right]$$

since $K = 0.5$ and $\mu_0 = 10$. Now $x_1 = 9.45$, so since $C_0^+ = C_0^- = 0$,

$$C_1^+ = \max[0, 9.45 - 10.5 + 0] = 0$$

Continued…

and

$$C_1^- = \max[0, 9.5 - 9.45 + 0] = 0.05$$

For period 2, we would use

$$C_2^+ = \max\left[0, x_2 - 10.5 + C_1^+\right]$$
$$= \max[0, x_2 - 10.5 + 0]$$

and

$$C_2^- = \max\left[0, 9.5 - x_2 + C_1^-\right]$$
$$= \max[0, 9.5 - x_2 + 0.05]$$

Since $x_2 = 7.99$, we obtain

$$C_2^+ = \max[0, 7.99 - 10.5 + 0] = 0$$

and

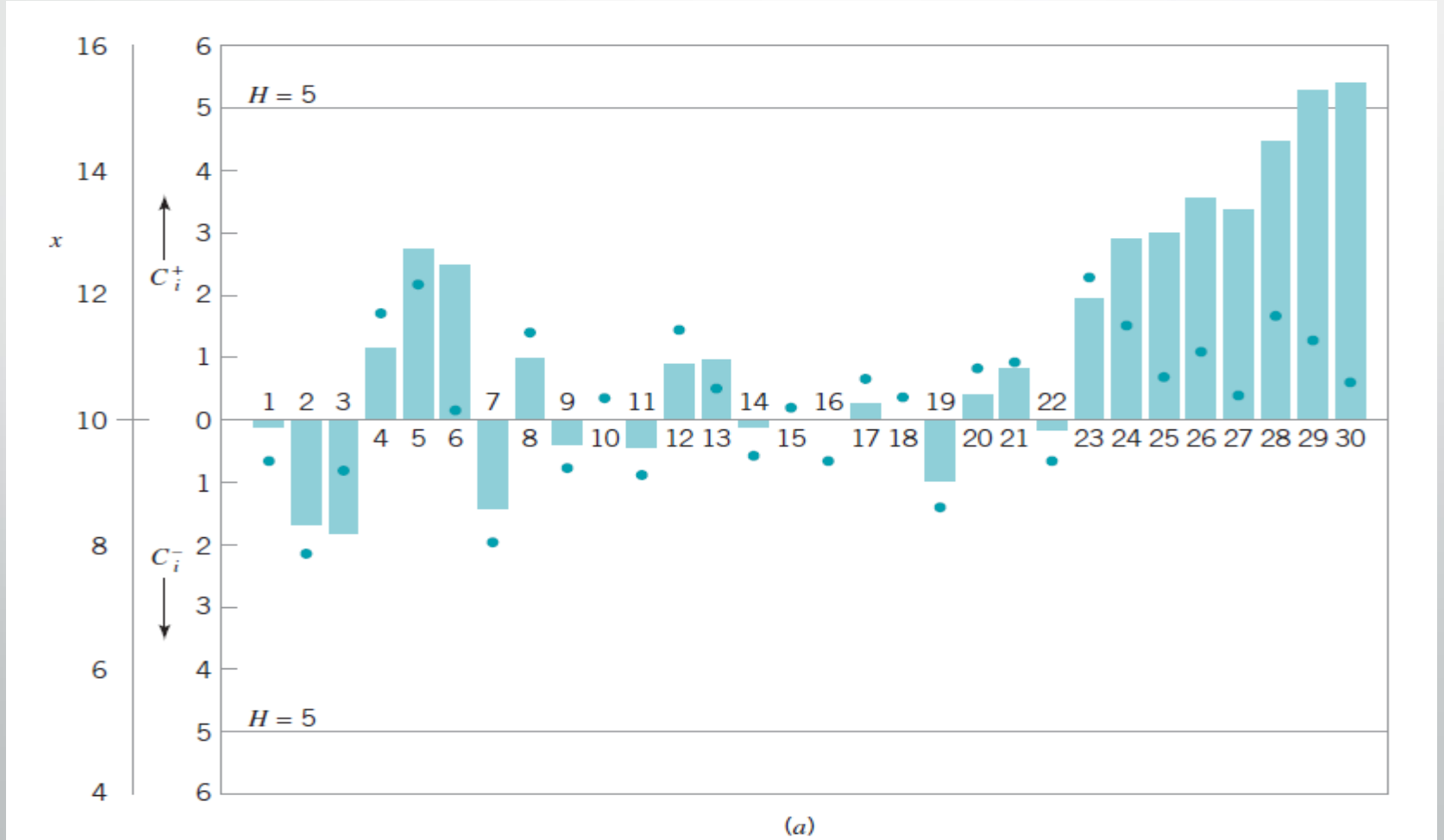
$$C_2^- = \max[0, 9.5 - 7.99 + 0.05] = 1.56$$

Conclusions from the chart

- The upperside cusum at period 29 is $C^+ 29 = 5.28$. Since this is the first period at which $C^+_i > H = 5$, we would conclude that the process is out of control at that point. The tabular cusum also indicates when the shift probably occurred.
- The counter N^+ records the number of consecutive periods since the upper-side cusum C^+_i rose above the value of zero. Since $N^+ = 7$ at period 29, we would conclude that the process was last in control at period $29 - 7 = 22$, so the shift likely occurred between periods 22 and 23.

CUSUM status chart

- It is useful to present a graphical display for the tabular cusum. These charts are sometimes called **cusum status charts.** They are constructed by plotting C^+_i and C^-_i versus the sample number.



Calculating new mean

- In situations where an adjustment to some manipulatable variable is required in order to bring the process back to the target value mo, it may be helpful to have an estimate of the new process mean following the shift. This can be computed from

$$\hat{\mu} = \begin{cases} \mu_0 + K + \dfrac{C_i^+}{N^+}, & \text{if } C_i^+ > H \\ \mu_0 - K - \dfrac{C_i^-}{N^-}, & \text{if } C_i^- > H \end{cases}$$

- Consider the cusum in period 29 with C^+_{29} =5.28

$$\begin{aligned} \hat{\mu} &= \mu_0 + K + \frac{C_{29}^+}{N^+} \\ &= 10.0 + 0.5 + \frac{5.28}{7} \\ &= 11.25 \end{aligned}$$

Recommendations for cusum

- The tabular cusum is designed by choosing values for the reference value K and the decision interval H. It is usually recommended that these parameters be selected to provide good average run length performance.

TABLE 9.3

ARL Performance of the Tabular Cusum with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

Continued…

- Define H = hσ and K = kσ, where σ is the standard deviation of the sample variable used in forming the cusum. Using h=4 or h=5 and k= 1 – 2 will generally provide a cusum that has good ARL properties against a shift of about 1σ in the process mean.
- K is chosen relative to the size of the shift that we want to detect, then we select h

Values of *k* and the Corresponding Values of *h* That Give $ARL_0 = 370$ for the Two-Sided Tabular Cusum [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

Calculating ARL

- Here we provide a simple formula for calculation
- For a **one-sided cusum** (that is, C^+_i or C^-_i) with parameters h and k,

$$\mathrm{ARL} = \frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}$$

- Where $\Delta = \delta^* - k$ for the upper one-sided cusum C^+_i, $\Delta = -\delta^* - k$ for the lower one-sided cusum C^-_i
  - $b=h + 1.166$, and $\delta^* = (\mu_1 - \mu_0)/\sigma$

Continued…

- To obtain the ARL of the two-sided cusum from the ARLs of the two one-sided statistics say, ARL^{+} and ARL^{-}, use:

$$\frac{1}{ARL} = \frac{1}{ARL^{+}} + \frac{1}{ARL^{-}}$$

To illustrate, consider the two-sided cusum with $k = \frac{1}{2}$ and $h = 5$. To find ARL_0 we would first calculate the ARL_0 values for the two-sided schemes—say, ARL_0^+ and ARL_0^-. Set $\delta^* = 0$; then $\Delta = \delta^* - k = 0 - \frac{1}{2} = -\frac{1}{2}$, $b = h + 1.166 = 5 + 1.166 = 6.166$, and from equation (9.6)

$$ARL_0^+ \simeq \frac{\exp\left[-2\left(-\frac{1}{2}\right)(6.166)\right] + 2\left(-\frac{1}{2}\right)(6.166) - 1}{2\left(-\frac{1}{2}\right)^2} = 938.2$$

By symmetry, we have $ARL_0^- = ARL_0^+$, and so from equation (9.7), the in-control ARL for the two-sided cusum is

$$\frac{1}{ARL_0} = \frac{1}{938.2} + \frac{1}{938.2}$$

or

$$ARL_0 = 469.1$$

Standardized Cusum

- Many users of the cusum prefer to standardize the variable x_i before performing the calculations.
- Let

$$y_i = \frac{x_i - \mu_0}{\sigma}$$

be the standardized value of x_i . Then:

The Standardized Two-Sided Cusum

$$C_i^+ = \max\left[0, y_i - k + C_{i-1}^+\right] \quad (9.9)$$

$$C_i^- = \max\left[0, -k - y_i + C_{i-1}^-\right] \quad (9.10)$$

Advantages of standardized CUSUM

- Many cusum charts can now have the same values of k and h, and the choices of these parameters are not scale dependent (that is, they do not depend on σ).
- Second, a standardized cusum leads naturally to a cusum for controlling variability,

The Fast Initial Response or Headstart Feature

- Aims to improve the sensitivity of a cusum at process start-up
- Increased sensitivity at process start-up would be desirable if the corrective action did not reset the mean to the target value.
- The fast initial response (FIR) or headstart essentially just sets the starting values C^+_0 and C^-_0 equal to some nonzero value, typically H/2. This is called a 50% headstart.

An example

- Consider the below data:

TABLE 9.6

A Cusum with a Headstart, Process Mean Equal to 100

		(a)			(b)		
Period i	x_i	$x_i - 103$	C_i^+	N^+	$97 - x_i$	C_i^-	N^-
1	102	–1	5	1	–5	1	1
2	97	–6	0	0	0	1	2
3	104	1	1	1	–7	0	0
4	93	–6	0	0	4	4	1
5	100	–3	0	0	–3	1	2
6	105	2	2	1	–8	0	0
7	96	–7	0	0	1	1	1
8	98	–5	0	0	–1	0	0
9	105	2	2	1	–8	0	0
10	99	–4	0	0	–2	0	0

Continued..

- These data have a target value of 100, K = 3, and H = 12. We will use a 50% headstart value of $C^+_0 = C^-_0 = H/2 = 6$.
- The first 10 samples are in control with mean equal to the target value of 100. Since $x_1 = 102$, the cusums for the first period will be :

$$
\begin{aligned}
C_i^+ &= \max\left[0, x_1 - 103 + C_0^+\right] \\
&= \max[0, 102 - 103 + 6] = 5 \\
C_i &= \max\left[0, 97 - x_1 + C_0^-\right] \\
&= \max[0, 97 - 102 + 6] = 1
\end{aligned}
$$

- The starting cusum value is the headstart H/2 = 6.
- From period 2 onward C^+_1 is unaffected by the headstart, and from period 3 onward C^-_1 is unaffected by the headstart. This has occurred because the process is in control at the target value of 100, and several consecutive observations near the target value were observed.

A Cusum for Monitoring Process Variability

- Let xi be the normally distributed process measurement with mean or target value mo and standard deviation s.
- The standardized value of x_i is

$$y_i = \frac{x_i - \mu_0}{\sigma}$$

- A new standardized quantity can be created:

$$v_i = \frac{\sqrt{|y_i|} - 0.822}{0.349} \qquad (9.11)$$

- Since the in-control distribution of vi is approximately N(0,1), two one-sided standardized scale (i.e., standard deviation) cusums can be established as follows(in the next slide).

Values of scale cusum

The Scale Cusum

$$S_i^+ = \max\left[0, v_i - k + S_{i-1}^+\right] \quad (9.12)$$

$$S_i^- = \max\left[0, -k - v_i + S_{i-1}^-\right] \quad (9.13)$$

- where $S^+_0 = S^-_0 = 0$ and the values of k and h are selected as in the cusum for controlling the process mean.
- The interpretation of the scale cusum is similar to the interpretation of the cusum for the mean. If the process standard deviation increases, the values of S^+_i will increase and eventually exceed h, whereas if the standard deviation decreases, the values of S^-_i will increase and eventually exceed h.

Rational Subgroup

- Although we have given the development of the tabular cusum for the case of individual observations (n = 1), it is easily extended to the case of averages of rational subgroups where the sample size n > 1. Simply replace x_i by x_i (the sample or subgroup average) in the above formulas, and replace σ with $\sigma/\sqrt{n}$.
- With Shewhart charts, the use of averages of rational subgroups substantially improves control chart performance. However, this does not always happen with the cusum.
  - Only if there is some significant economy of scale or some other valid reason for taking samples of size greater than unity should one consider using n > 1 with the cusum.
  - One practical reason for using rational subgroups of size n>1 is that we could now set up a cusum on the sample variance and use it to monitor process variability.